

Symmetries of curved superspace

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Abstract

The formalism to determine (conformal) isometries of a given curved superspace was elaborated almost two decades ago in the context of the old minimal formulation for $\mathcal{N} = 1$ supergravity in four dimensions (4D). This formalism is universal, for it may readily be generalized to supersymmetric backgrounds associated with any supergravity theory formulated in superspace. In particular, it has already been used to construct rigid supersymmetric field theories in 5D $\mathcal{N} = 1$, 4D $\mathcal{N} = 2$ and 3D (p, q) anti-de Sitter superspaces. In the last two years, there have appeared a number of publications devoted to the construction of supersymmetric backgrounds in off-shell 4D $\mathcal{N} = 1$ supergravity theories using component field considerations. Here we demonstrate how to read off the key results of these recent publications from the more general superspace approach developed in the 1990s. We also present a universal superspace setting to construct supersymmetric backgrounds, which is applicable to any of the known off-shell formulations for $\mathcal{N} = 1$ supergravity. This approach is based on the realizations of the new minimal and non-minimal supergravity theories as super-Weyl invariant couplings of the old minimal supergravity to certain conformal compensators.

1 Introduction

With the motivation to elaborate supersymmetric quantum field theory in curved space, almost two decades ago a formalism was developed [1] to determine (conformal) isometries of a given curved superspace originating within the old minimal formulation for $\mathcal{N} = 1$ supergravity in four dimensions (4D).¹ As simple illustrations of the formalism, it was used in [1] to compute (i) the conformal Killing supervector fields of any conformally flat $\mathcal{N} = 1$ superspace; and (ii) the Killing supervector fields of $\mathcal{N} = 1$ AdS superspace. The approach presented in [1] is universal, for in principle it may be generalized to supersymmetric backgrounds associated with any supergravity theory formulated in superspace. In particular, it has already been used to construct rigid supersymmetric field theories in 5D $\mathcal{N} = 1$ [9], 4D $\mathcal{N} = 2$ [10, 11] and 3D (p, q) anti-de Sitter [12, 13, 14] superspaces.

Recently, numerous publications have appeared devoted to the construction of supersymmetric backgrounds associated with the old minimal and the new minimal supergravities using component field considerations [15, 16, 17, 18, 19, 20, 21, 22, 23]. These backgrounds are simply curved (pseudo) Riemannian spaces that allow unbroken rigid supersymmetries. The techniques used in these publications make no use of the superspace formalism of [1]. However, since the rigid supersymmetry transformations are special isometry transformations of a given curved superspace, there should exist a simple procedure to derive the key component results of [15, 16, 17, 18, 19, 20, 21, 22, 23] from the more general superspace construction of [1], for the latter allows one to determine all the isometries. One of the goals of the present note is to work out such a procedure. Our second, more important goal is to present a universal superspace setting, which can be used for any of the known off-shell formulations for $\mathcal{N} = 1$ supergravity, to determine the isometries of curved backgrounds. This novel approach can immediately be generalized to all known off-shell supergravities in diverse dimensions, including the important cases of 3D $\mathcal{N} \leq 4$, 4D $\mathcal{N} = 2$ and 5D $\mathcal{N} = 1$ supergravity theories.

It should be mentioned that a considerable amount of the results in [15, 16, 17, 18, 19, 21, 22, 23] are devoted to supersymmetric backgrounds with Euclidean signature. Our analysis is restricted to curved space-times allowing unbroken supersymmetry.

¹Historically, this supergravity formulation was first constructed by Wess and Zumino in superspace [2] (see also [3]), and soon after it was independently developed using the component tensor calculus [4]. The superspace [2] and the component [4] approaches to old minimal supergravity are equivalent, for the latter can readily be deduced from the former [5, 6, 7] (see [8] for a review).

This paper is organized as follows. In section 2 we briefly review the superspace geometry of old minimal supergravity following the notation and conventions adopted in [1]. Section 3 contains a summary of the main properties of the (conformal) Killing supervector fields of a curved superspace derived in [1]. In section 4 we study those supergravity backgrounds without covariant fermionic fields which allow some unbroken (conformal) supersymmetries. Section 5 describes a universal superspace setting to construct supersymmetric backgrounds, which is applicable to any of the known off-shell formulations for $\mathcal{N} = 1$ supergravity. Concluding comments are given in section 6.

2 The Wess-Zumino superspace geometry

In describing the Wess-Zumino superspace geometry (see [8] for a review), we follow the notation and conventions of [1].² In particular, the coordinates of $\mathcal{N} = 1$ curved superspace \mathcal{M} are denoted $z^M = (x^m, \theta^\mu, \bar{\theta}_{\dot{\mu}})$. The superspace geometry is described by covariant derivatives of the form

$$\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}^{\dot{\alpha}}) = E_A + \Omega_A . \quad (2.1)$$

Here E_A denotes the inverse vielbein, $E_A = E_A^M \partial_M$, and Ω_A the Lorentz connection,

$$\Omega_A = \frac{1}{2} \Omega_A^{bc} M_{bc} = \Omega_A^{\beta\gamma} M_{\beta\gamma} + \Omega_A^{\dot{\beta}\dot{\gamma}} \bar{M}_{\dot{\beta}\dot{\gamma}} , \quad (2.2)$$

with $M_{bc} \Leftrightarrow (M_{\beta\gamma}, \bar{M}_{\dot{\beta}\dot{\gamma}})$ the Lorentz generators. The covariant derivatives obey the following anti-commutation relations:

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}\} = -2i\mathcal{D}_{\alpha\dot{\alpha}} ,$$

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = -4\bar{R}M_{\alpha\beta} , \quad \{\bar{\mathcal{D}}_{\dot{\alpha}}, \bar{\mathcal{D}}_{\dot{\beta}}\} = 4R\bar{M}_{\dot{\alpha}\dot{\beta}} , \quad (2.3a)$$

$$[\bar{\mathcal{D}}_{\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}] = -i\varepsilon_{\dot{\alpha}\dot{\beta}} \left(R\mathcal{D}_\beta + G_{\beta}^{\dot{\gamma}} \bar{\mathcal{D}}_{\dot{\gamma}} - (\bar{\mathcal{D}}^{\dot{\gamma}} G_{\beta}^{\dot{\delta}}) \bar{M}_{\dot{\gamma}\dot{\delta}} + 2W_{\beta}^{\gamma\delta} M_{\gamma\delta} \right) - i(\mathcal{D}_\beta R) \bar{M}_{\dot{\alpha}\dot{\beta}} , \quad (2.3b)$$

$$[\mathcal{D}_\alpha, \mathcal{D}_{\beta\dot{\beta}}] = i\varepsilon_{\alpha\beta} \left(\bar{R}\bar{\mathcal{D}}_{\dot{\beta}} + G_{\dot{\beta}}^{\gamma} \mathcal{D}_\gamma - (\mathcal{D}^\gamma G_{\dot{\beta}}^{\delta}) M_{\gamma\delta} + 2\bar{W}_{\dot{\beta}}^{\dot{\gamma}\dot{\delta}} \bar{M}_{\dot{\gamma}\dot{\delta}} \right) + i(\bar{\mathcal{D}}_{\dot{\beta}} \bar{R}) M_{\alpha\beta} , \quad (2.3c)$$

$$[\mathcal{D}_{\alpha\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}] = \varepsilon_{\dot{\alpha}\dot{\beta}} \psi_{\alpha\beta} + \varepsilon_{\alpha\beta} \psi_{\dot{\alpha}\dot{\beta}} , \quad (2.3d)$$

²These conventions are nearly identical to those of Wess and Bagger [8]. To convert the notation of [1] to that of [8], one replaces $R \rightarrow 2R$, $G_{\alpha\dot{\alpha}} \rightarrow 2G_{\alpha\dot{\alpha}}$, and $W_{\alpha\beta\gamma} \rightarrow 2W_{\alpha\beta\gamma}$. In addition, the vector derivative is defined by $\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}\} = -2i\mathcal{D}_{\alpha\dot{\alpha}}$. Finally, the spinor Lorentz generators $(\sigma_{ab})_{\alpha}^{\beta}$ and $(\tilde{\sigma}_{ab})^{\dot{\alpha}}_{\dot{\beta}}$ used in [1] have an extra minus sign as compared with [8], specifically $\sigma_{ab} = -\frac{1}{4}(\sigma_a \tilde{\sigma}_b - \sigma_b \tilde{\sigma}_a)$ and $\tilde{\sigma}_{ab} = -\frac{1}{4}(\tilde{\sigma}_a \sigma_b - \tilde{\sigma}_b \sigma_a)$.

where

$$\begin{aligned} \psi_{\alpha\beta} := & -iG_{(\alpha}{}^{\dot{\gamma}}\mathcal{D}_{\beta)\dot{\gamma}} + \frac{1}{2}(\mathcal{D}_{(\alpha}R)\mathcal{D}_{\beta)} + \frac{1}{2}(\mathcal{D}_{(\alpha}G_{\beta)}{}^{\dot{\gamma}})\bar{\mathcal{D}}_{\dot{\gamma}} + W_{\alpha\beta}{}^{\gamma}\mathcal{D}_{\gamma} \\ & + \frac{1}{4}((\bar{\mathcal{D}}^2 - 8R)\bar{R})M_{\alpha\beta} + (\mathcal{D}_{(\alpha}W_{\beta)}{}^{\gamma\delta})M_{\gamma\delta} - \frac{1}{2}(\mathcal{D}_{(\alpha}\bar{\mathcal{D}}^{\dot{\gamma}}G_{\beta)}{}^{\dot{\delta}})\bar{M}_{\dot{\gamma}\dot{\delta}} , \end{aligned} \quad (2.3e)$$

$$\begin{aligned} \psi_{\dot{\alpha}\dot{\beta}} := & -iG_{\gamma(\dot{\alpha}}\mathcal{D}^{\gamma}{}_{\dot{\beta})} - \frac{1}{2}(\bar{\mathcal{D}}_{(\dot{\alpha}}\bar{R})\bar{\mathcal{D}}_{\dot{\beta})} - \frac{1}{2}(\bar{\mathcal{D}}_{(\dot{\alpha}}G^{\gamma}{}_{\dot{\beta})})\mathcal{D}_{\gamma} - \bar{W}_{\dot{\alpha}\dot{\beta}}{}^{\dot{\gamma}}\bar{\mathcal{D}}_{\dot{\gamma}} \\ & + \frac{1}{4}((\mathcal{D}^2 - 8\bar{R})R)\bar{M}_{\dot{\alpha}\dot{\beta}} - (\bar{\mathcal{D}}_{(\dot{\alpha}}\bar{W}_{\dot{\beta})}{}^{\dot{\gamma}\dot{\delta}})\bar{M}_{\dot{\gamma}\dot{\delta}} + \frac{1}{2}(\bar{\mathcal{D}}_{(\dot{\alpha}}\mathcal{D}^{\gamma}G^{\delta}{}_{\dot{\beta})})M_{\gamma\delta} . \end{aligned} \quad (2.3f)$$

The torsion tensors R , $G_a = \bar{G}_a$ and $W_{\alpha\beta\gamma} = W_{(\alpha\beta\gamma)}$ satisfy the Bianchi identities

$$\bar{\mathcal{D}}_{\dot{\alpha}}R = 0 , \quad \bar{\mathcal{D}}_{\dot{\alpha}}W_{\alpha\beta\gamma} = 0 , \quad (2.4a)$$

$$\bar{\mathcal{D}}^{\dot{\gamma}}G_{\alpha\dot{\gamma}} = \mathcal{D}_{\alpha}R , \quad (2.4b)$$

$$\mathcal{D}^{\gamma}W_{\alpha\beta\gamma} = i\mathcal{D}_{(\alpha}{}^{\dot{\gamma}}G_{\beta)\dot{\gamma}} . \quad (2.4c)$$

A supergravity gauge transformation is defined to act on the covariant derivatives and any tensor superfield U (with its indices suppressed) by the rule

$$\delta_{\mathcal{K}}\mathcal{D}_A = [\mathcal{K}, \mathcal{D}_A] , \quad \delta_{\mathcal{K}}U = \mathcal{K}U . \quad (2.5a)$$

Here the gauge parameter \mathcal{K} has the explicit form

$$\mathcal{K} = \xi^B\mathcal{D}_B + K^{\gamma\delta}M_{\gamma\delta} + \bar{K}^{\dot{\gamma}\dot{\delta}}\bar{M}_{\dot{\gamma}\dot{\delta}} = \bar{\mathcal{K}} \quad (2.5b)$$

and describes a general coordinate transformation generated by the supervector field ξ^B as well as a local Lorentz transformation generated by the symmetric spinor $K^{\gamma\delta} + \xi^B\Omega_B{}^{\gamma\delta}$ and its conjugate.

The algebra of covariant derivatives (2.3) is invariant under super-Weyl transformations [24]

$$\delta_{\sigma}\mathcal{D}_{\alpha} = \left(\frac{1}{2}\sigma - \bar{\sigma}\right)\mathcal{D}_{\alpha} - (\mathcal{D}^{\beta}\sigma)M_{\alpha\beta} , \quad (2.6a)$$

$$\delta_{\sigma}\bar{\mathcal{D}}_{\dot{\alpha}} = \left(\frac{1}{2}\bar{\sigma} - \sigma\right)\bar{\mathcal{D}}_{\dot{\alpha}} - (\bar{\mathcal{D}}^{\dot{\beta}}\bar{\sigma})\bar{M}_{\dot{\alpha}\dot{\beta}} , \quad (2.6b)$$

$$\begin{aligned} \delta_{\sigma}\mathcal{D}_{\alpha\dot{\alpha}} = & -\frac{1}{2}(\sigma + \bar{\sigma})\mathcal{D}_{\alpha\dot{\alpha}} - \frac{i}{2}(\bar{\mathcal{D}}_{\dot{\alpha}}\bar{\sigma})\mathcal{D}_{\alpha} - \frac{i}{2}(\mathcal{D}_{\alpha}\sigma)\bar{\mathcal{D}}_{\dot{\alpha}} \\ & - (\mathcal{D}^{\beta}{}_{\dot{\alpha}}\sigma)M_{\alpha\beta} - (\mathcal{D}_{\alpha}{}^{\dot{\beta}}\bar{\sigma})\bar{M}_{\dot{\alpha}\dot{\beta}} , \end{aligned} \quad (2.6c)$$

with the scalar parameter σ being covariantly chiral,

$$\bar{\mathcal{D}}_{\dot{\alpha}}\sigma = 0 , \quad (2.7)$$

provided the torsion tensors transform³ as follows:

$$\delta_\sigma R = -2\sigma R - \frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\bar{\sigma} , \quad (2.8a)$$

$$\delta_\sigma G_{\alpha\dot{\alpha}} = -\frac{1}{2}(\sigma + \bar{\sigma})G_{\alpha\dot{\alpha}} + i\mathcal{D}_{\alpha\dot{\alpha}}(\bar{\sigma} - \sigma) , \quad (2.8b)$$

$$\delta_\sigma W_{\alpha\beta\gamma} = -\frac{3}{2}\sigma W_{\alpha\beta\gamma} . \quad (2.8c)$$

Let $\mathfrak{D}_A = (\mathfrak{D}_a, \mathfrak{D}_\alpha, \bar{\mathfrak{D}}^{\dot{\alpha}})$ be another set of superspace covariant derivatives which describe a curved supergravity background. The two superspace geometries, which are associated with \mathcal{D}_A and \mathfrak{D}_A , are said to be conformally related if their covariant derivatives are related by a finite super-Weyl transformation

$$\mathfrak{D}_\alpha = e^{\frac{1}{2}\omega - \bar{\omega}} \left(\mathcal{D}_\alpha - (\mathcal{D}^\beta \omega) M_{\alpha\beta} \right) , \quad (2.9a)$$

$$\bar{\mathfrak{D}}_{\dot{\alpha}} = e^{\frac{1}{2}\bar{\omega} - \omega} \left(\bar{\mathcal{D}}_{\dot{\alpha}} - (\bar{\mathcal{D}}^{\dot{\beta}} \bar{\omega}) \bar{M}_{\dot{\alpha}\dot{\beta}} \right) , \quad (2.9b)$$

$$\mathfrak{D}_{\alpha\dot{\alpha}} = \frac{i}{2} \{ \mathfrak{D}_\alpha, \bar{\mathfrak{D}}_{\dot{\alpha}} \} , \quad (2.9c)$$

where ω is a covariantly chiral scalar, $\bar{\mathcal{D}}_{\dot{\alpha}}\omega = 0$.

3 (Conformal) Killing supervector fields

Let us fix a curved background superspace \mathcal{M} . In accordance with [1], a supervector field $\xi = \xi^B E_B$ on \mathcal{M} is called conformal Killing if there exists a symmetric spinor $K^{\gamma\delta}$ and a covariantly chiral scalar σ such that

$$(\delta_K + \delta_\sigma)\mathcal{D}_A = 0 . \quad (3.1)$$

In other words, the coordinate transformation generated by ξ can be accompanied by certain Lorentz and super-Weyl transformations such that the superspace geometry does not change.

As demonstrated in [1], all information about the conformal Killing supervector field

³The super-Weyl transformation of $G_{\alpha\dot{\alpha}}$ given in [1], eq. (5.5.14), contains a typo.

is encoded in the special case of eq. (3.1) with $A = \alpha$. Making use of the variation

$$\begin{aligned}
\delta_{\mathcal{K}} \mathcal{D}_{\alpha} = & \left(K_{\alpha}^{\beta} - \mathcal{D}_{\alpha} \xi^{\beta} - \frac{i}{2} \xi_{\alpha\dot{\beta}} G^{\beta\dot{\beta}} \right) \mathcal{D}_{\beta} + \left(\mathcal{D}_{\alpha} \bar{\xi}^{\dot{\beta}} + \frac{i}{2} \xi_{\alpha}^{\dot{\beta}} \bar{R} \right) \bar{\mathcal{D}}_{\dot{\beta}} \\
& + 2i \left(\delta_{\alpha}^{\beta} \bar{\xi}^{\dot{\beta}} - \frac{i}{4} \mathcal{D}_{\alpha} \xi^{\beta\dot{\beta}} \right) \mathcal{D}_{\beta\dot{\beta}} \\
& - \left(\mathcal{D}_{\alpha} K^{\beta\gamma} + 4\delta_{\alpha}^{(\beta} \xi^{\gamma)} \bar{R} - \frac{i}{2} \delta_{\alpha}^{(\beta} \xi^{\gamma)\dot{\gamma}} \bar{\mathcal{D}}_{\dot{\gamma}} \bar{R} - \frac{i}{2} \xi_{\alpha\dot{\alpha}} \mathcal{D}^{(\beta} G^{\gamma)\dot{\alpha}} \right) M_{\beta\gamma} \\
& - (\mathcal{D}_{\alpha} \bar{K}^{\dot{\beta}\dot{\gamma}} + i \xi_{\alpha\dot{\alpha}} \bar{W}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}) \bar{M}_{\dot{\beta}\dot{\gamma}} ,
\end{aligned} \tag{3.2}$$

in conjunction with the super-Weyl transformation (2.6), we obtain a number of conditions on the parameters which can be split in two groups. The first group consists of the following equations

$$\delta_{\alpha}^{\beta} \bar{\xi}^{\dot{\beta}} = \frac{i}{4} \mathcal{D}_{\alpha} \xi^{\beta\dot{\beta}} \quad \Rightarrow \quad \bar{\xi}^{\dot{\alpha}} = \frac{i}{8} \mathcal{D}_{\alpha} \xi^{\alpha\dot{\alpha}} , \tag{3.3a}$$

$$K_{\alpha\beta} = \mathcal{D}_{(\alpha} \xi_{\beta)} - \frac{i}{2} \xi_{(\alpha}^{\dot{\beta}} G_{\beta)\dot{\beta}} , \tag{3.3b}$$

$$\sigma = \frac{1}{3} (\mathcal{D}^{\alpha} \xi_{\alpha} + 2 \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} - i \xi^a G_a) , \tag{3.3c}$$

and their conjugates. Eq. (3.3c) has to be taken in conjunction with the chirality condition, eq. (2.7), obeyed by the super-Weyl parameter. The meaning of the relations (3.3) is that the parameters ξ^{α} , $K^{\alpha\beta}$ and σ are completely determined in terms of the real vector ξ^a and its covariant derivatives. This is why we may also use the notation $\mathcal{K} = \mathcal{K}[\xi]$, and similarly for the Lorentz and super-Weyl parameters, e.g. $\sigma = \sigma[\xi]$.

The second group comprises the following equations and their conjugates:

$$\mathcal{D}_{\alpha} \bar{\xi}_{\dot{\alpha}} = -\frac{i}{2} \xi_{\alpha\dot{\alpha}} \bar{R} , \tag{3.4a}$$

$$\bar{\mathcal{D}}_{\dot{\alpha}} K^{\beta\gamma} = i \xi_{\alpha\dot{\alpha}} W^{\alpha\beta\gamma} , \tag{3.4b}$$

$$\mathcal{D}_{\alpha} K^{\beta\gamma} = -\delta_{\alpha}^{(\beta} \mathcal{D}^{\gamma)} \sigma - 4\delta_{\alpha}^{(\beta} \xi^{\gamma)} \bar{R} + \frac{i}{2} \delta_{\alpha}^{(\beta} \xi^{\gamma)\dot{\gamma}} \bar{\mathcal{D}}_{\dot{\gamma}} \bar{R} + \frac{i}{2} \xi_{\alpha\dot{\alpha}} \mathcal{D}^{(\beta} G^{\gamma)\dot{\alpha}} . \tag{3.4c}$$

These relations allow us to express multiple covariant spinor derivatives of the parameters in terms of the parameters.⁴

Since the real vector ξ^a is the only independent parameter, there should exist a closed-form equation obeyed by ξ^a . It has the form

$$\mathcal{D}_{(\alpha} \xi_{\beta)\dot{\beta}} = 0 . \tag{3.5}$$

⁴In the non-conformal case, which corresponds to $\sigma = 0$, the first spinor covariant derivatives of the parameters ξ^B , $K^{\beta\gamma}$ and $\bar{K}^{\dot{\beta}\dot{\gamma}}$ are linear combinations of these parameters.

Simple corollaries of this equation⁵ include the linearity condition

$$(\mathcal{D}^2 + 2\bar{R})\xi_a = 0 , \quad (3.6)$$

and the conformal Killing equation

$$\mathcal{D}_a \xi_b + \mathcal{D}_b \xi_a = \frac{1}{2} \eta_{ab} \mathcal{D}^c \xi_c . \quad (3.7)$$

As shown in [1], all information about the conformal Killing supervector field is encoded in the master equation (3.5). Specifically, if this equation holds and the definitions (3.3) are adopted, then the consistency conditions (3.4) are identically satisfied, and the super-Weyl parameter $\sigma[\xi]$ is covariantly chiral. As a result, an alternative definition of the conformal Killing supervector field can be given. It is a real supervector field

$$\xi = \xi^A E_A , \quad \xi^A = \left(\xi^a, -\frac{i}{8} \bar{\mathcal{D}}_{\dot{\beta}} \xi^{\alpha \dot{\beta}}, -\frac{i}{8} \mathcal{D}^{\beta} \xi_{\beta \dot{\alpha}} \right) \quad (3.8)$$

which obeys the master equation (3.5).

If ξ_1 and ξ_2 are two conformal Killing supervector fields, their Lie bracket $[\xi_1, \xi_2]$ is a conformal Killing supervector field [1]. It is obvious that, for any real c -numbers r_1 and r_2 , the linear combination $r_1 \xi_1 + r_2 \xi_2$ is a conformal Killing supervector field. Thus the set of all conformal Killing supervector fields is a super Lie algebra. The conformal Killing supervector fields generate symmetries of a super-Weyl invariant field theory on \mathcal{M} .

We need to recall one more result from [1]. Suppose we have another curved superspace \mathfrak{M} that is conformally related to \mathcal{M} . This means that the covariant derivatives \mathcal{D}_A and \mathfrak{D}_A , which correspond to \mathcal{M} and \mathfrak{M} respectively, are related to each other according to the rule (2.9). It turns out that the two superspaces \mathcal{M} and \mathfrak{M} have the same conformal Killing supervector fields. Given such a supervector field ξ , it can be represented in two different forms

$$\xi = \xi^A E_A = \boldsymbol{\xi}^A \mathfrak{E}_A , \quad (3.9)$$

where \mathfrak{E}_A is the inverse vielbein associated with the covariant derivatives \mathfrak{D}_A . Then the super-Weyl parameter $\sigma[\xi]$ and $\sigma[\boldsymbol{\xi}]$ are related to each other as follows

$$\sigma[\boldsymbol{\xi}] = \sigma[\xi] - \xi \omega . \quad (3.10)$$

The derivation of this result is given in [1].

⁵The equation (3.5) is analogous to the conformal Killing equation, $\nabla_{(\alpha} (\dot{\alpha} V_{\beta)}^{\dot{\beta}}) = 0$, on a (pseudo) Riemannian four-dimensional manifold.

A Killing supervector field ξ on \mathcal{M} is a conformal Killing supervector field with the additional property $\sigma[\xi] = 0$, or equivalently

$$\delta_{\mathcal{K}}\mathcal{D}_A = [\mathcal{K}, \mathcal{D}_A] = 0 . \quad (3.11)$$

The condition that the super-Weyl parameter (3.3c) be equal to zero is

$$\mathcal{D}^\alpha \xi_\alpha = -iG^a \xi_a . \quad (3.12)$$

If ξ_1 and ξ_2 are Killing supervector fields, their Lie bracket $[\xi_1, \xi_2]$ is a Killing supervector field. Thus the set of all Killing supervector fields forms a super Lie algebra. The Killing supervector fields generate the isometries of \mathcal{M} , and symmetries of a field theory on \mathcal{M} .

To study supersymmetry transformations at the component level, it is useful to spell out one of the implications of eq. (2.6) with $A = a$. Specifically, we consider the equation $(\delta_{\mathcal{K}} + \delta_\sigma)\mathcal{D}_{\alpha\dot{\alpha}} = 0$ and read off its part proportional to a linear combination of the spinor covariant derivatives \mathcal{D}_β . The results is

$$\begin{aligned} 0 = \mathcal{D}_{\alpha\dot{\alpha}}\xi_\beta &- \frac{i}{2}\varepsilon_{\alpha\beta}\bar{\mathcal{D}}_{\dot{\alpha}}\bar{\sigma} + i\xi_\alpha G_{\beta\dot{\alpha}} - i\varepsilon_{\alpha\beta}\bar{\xi}_{\dot{\alpha}}R \\ &- \frac{1}{4}\xi_{\beta\dot{\alpha}}\mathcal{D}_\alpha R - \frac{1}{2}\xi^{\gamma\dot{\gamma}}W_{\alpha\beta\gamma} + \frac{1}{4}\xi_\alpha{}^{\dot{\gamma}}\bar{\mathcal{D}}_{\dot{\alpha}}G_{\beta\dot{\gamma}} . \end{aligned} \quad (3.13)$$

In the case of isometry transformations on \mathcal{M} , we have to set $\sigma = 0$. Eq. (3.13) will play a fundamental role in our subsequent analysis.

4 Supersymmetric backgrounds

We wish to look for those curved backgrounds which admit some unbroken (conformal) supersymmetries. By definition, such a superspace possesses a (conformal) Killing supervector field ξ^A with the property

$$\xi^a| = 0 , \quad \epsilon^\alpha := \xi^\alpha| \neq 0 , \quad (4.1)$$

where $U|$ denotes the $\theta, \bar{\theta}$ independent part of a tensor superfield $U(z) = U(x^m, \theta^\mu, \bar{\theta}_{\dot{\mu}})$,

$$U| := U|_{\theta^\mu = \bar{\theta}_{\dot{\mu}} = 0} . \quad (4.2)$$

We will refer to the field $U|$ as the bar-projection of U . Our analysis will be restricted to supergravity backgrounds without covariant fermionic fields, that is

$$\mathcal{D}_\alpha R| = 0 , \quad \mathcal{D}_\alpha G_{\beta\dot{\beta}}| = 0 , \quad W_{\alpha\beta\gamma}| = 0 . \quad (4.3)$$

This means that the gravitino can completely be gauged away such that the bar-projection of the vector covariant derivative⁶ is

$$\tilde{\nabla}_a := \mathcal{D}_a| = \nabla_a + \frac{1}{6}\varepsilon_{abcd}b^b M^{cd}, \quad \nabla_a = e_a^m \partial_m + \frac{1}{2}\omega_a^{cd} M_{cd}, \quad (4.4)$$

where ∇_a denotes the ordinary torsion-free covariant derivative,

$$[\nabla_a, \nabla_b] = \frac{1}{2}\mathcal{R}_{ab}{}^{cd} M_{cd}, \quad (4.5)$$

and the vector field b_a is one of the auxiliary fields M , \bar{M} and b_a , which correspond to the old minimal supergravity and are defined as⁷

$$R| = -\frac{1}{3}M, \quad G_a| = -\frac{2}{3}b_a. \quad (4.6)$$

4.1 Conformal supersymmetry

Let us first determine the conditions for unbroken conformal supersymmetry. For this, we consider the $\theta, \bar{\theta}$ independent part of the equation (3.13). With the definition

$$\mathcal{D}_\alpha \sigma| = -\frac{2}{3}\zeta_\alpha, \quad (4.7)$$

the result is

$$2\nabla_a \epsilon_\beta - \frac{i}{3}\left\{(\sigma_a \bar{\zeta})_\beta + (\sigma_a \bar{\epsilon})_\beta M - 2(\sigma_{ac} \epsilon)_\beta b^c + 2b_a \epsilon_\beta\right\} = 0. \quad (4.8)$$

This equation allows one to express the conformal spinor parameter $\bar{\zeta}_{\dot{\alpha}}$ in terms of ϵ_α and its conjugate.

Eq. (4.8) can be rewritten in a different and more illuminating form if we introduce the first-order operator

$$\mathfrak{D}_a \epsilon_\beta := (\nabla_a - \frac{i}{2}b_a)\epsilon_\beta \quad (4.9)$$

which can be viewed as a U(1) gauge covariant derivative. Then one may see that (4.8) is equivalent to

$$2\mathfrak{D}_a \epsilon_\beta - \frac{i}{3}\left\{(\sigma_a \bar{\zeta})_\beta + (\sigma_a \bar{\epsilon})_\beta M + (\sigma_a \tilde{\sigma}_c \epsilon)_\beta b^c\right\} = 0. \quad (4.10)$$

⁶The bar-projection of a covariant derivative, $\mathcal{D}_A|$, is defined by the rule $(\mathcal{D}_A|U)| := (\mathcal{D}_A U)|$, for any tensor superfield U . The bar-projection of a product of several covariant derivatives, $\mathcal{D}_{A_1} \cdots \mathcal{D}_{A_n}|$, is defined similarly.

⁷To simplify comparison with the results of [15], here we make use of the same definition of the auxiliary fields following [8]. These are related to the supergravity auxiliary fields \mathbb{B} and \mathbb{A}_a used [1] by the rule: $M = -\mathbb{B}$ and $b_a = -2\mathbb{A}_a$.

Expressing here $\bar{\zeta}_{\dot{\alpha}}$ in terms of ϵ_{α} and its conjugate leads to the equation

$$\mathfrak{D}_a \epsilon_{\beta} + \frac{1}{4}(\sigma_a \tilde{\sigma}^c \mathfrak{D}_c \epsilon)_{\beta} = 0 , \quad (4.11)$$

which is equivalent to

$$\mathfrak{D}_{\alpha\dot{\gamma}} \epsilon_{\beta} + \mathfrak{D}_{\beta\dot{\gamma}} \epsilon_{\alpha} = 0 . \quad (4.12)$$

We conclude that ϵ_{β} is a charged conformal Killing spinor. Given a non-zero solution $\epsilon_{\alpha}(x)$ of (4.12), as well as a non-zero complex number $z \in \mathbb{C}$, it is obvious that $z \epsilon_{\alpha}(x)$ is also a solution of the same equation. We conclude that the minimal amount of conformal supersymmetry is two supercharges, which agrees with the conclusions in [20].

If ϵ_{β} is a commuting conformal Killing spinor obeying the equation (4.12), the *null* vector $\mathcal{V}_{\beta\dot{\beta}} := \epsilon_{\beta} \bar{\epsilon}_{\dot{\beta}}$ is a conformal Killing vector field,

$$\nabla_{(\alpha} (\dot{\alpha} \mathcal{V}_{\beta)}^{\dot{\beta}}) = 0 . \quad (4.13)$$

Thus we have re-derived one of the key results of [20].⁸

4.2 Rigid supersymmetry

In the non-conformal case, setting $\bar{\zeta}^{\dot{\alpha}} = 0$ in (4.8) gives the equation for unbroken rigid supersymmetry

$$2\nabla_a \epsilon_{\beta} - \frac{i}{3} \left\{ (\sigma_a \bar{\epsilon})_{\beta} M - 2(\sigma_{ac} \epsilon)_{\beta} b^c + 2b_a \epsilon_{\beta} \right\} = 0 . \quad (4.14)$$

This equation coincides with that given in [15] keeping in mind the fact that the matrices σ_{ab} used in [15] differ in sign from ours.

4.3 Curved spacetimes admitting four supercharges

We now turn to deriving those conditions on the background geometry which guarantee that the spacetime under consideration possesses nontrivial solutions of eq. (4.14) giving rise to exactly four supercharges. The main idea of our analysis below is that the conditions (4.3) must be supersymmetric.

⁸It was demonstrated in [20] that \mathcal{M} possesses a conformal Killing spinor if and only if it has a null conformal Killing vector.

To start with, consider the identity

$$\begin{aligned} 0 &= \delta_{\mathcal{K}} \mathcal{D}_{\alpha} R = \xi^D \mathcal{D}_D \mathcal{D}_{\alpha} + K_{\alpha}^{\gamma} \mathcal{D}_{\gamma} R \\ &= \xi^c \mathcal{D}_c \mathcal{D}_{\alpha} R - \frac{1}{2} \xi_{\alpha} \mathcal{D}^2 R + 2i \bar{\xi}^{\dot{\gamma}} \mathcal{D}_{\alpha \dot{\gamma}} R + K_{\alpha}^{\gamma} \mathcal{D}_{\gamma} R . \end{aligned} \quad (4.15)$$

The bar-projection of this relation is

$$\epsilon_{\alpha} \mathcal{D}^2 R| - 4i \bar{\epsilon}^{\dot{\gamma}} \nabla_{\alpha \dot{\gamma}} R| = 0 . \quad (4.16)$$

This is equivalent to

$$\mathcal{D}^2 R| = 0 , \quad (4.17a)$$

$$\nabla_a M = 0 . \quad (4.17b)$$

The complete expression for $\mathcal{D}^2 R|$ in terms of the supergravity fields can be found in, e.g., [1] and [8]. We will not need this expression, for the condition (4.17a) proves to follow from a more general result to be derived shortly. Eq. (4.17b) means that M is a constant parameter.

The next condition to analyze is

$$0 = \delta_{\mathcal{K}} \mathcal{D}_{\alpha} G_{\beta \dot{\beta}} = \xi^D \mathcal{D}_D \mathcal{D}_{\alpha} + K_{\alpha}^{\delta} \mathcal{D}_{\delta} G_{\beta \dot{\beta}} + K_{\beta}^{\delta} \mathcal{D}_{\alpha} G_{\delta \dot{\beta}} + \bar{K}_{\dot{\beta}}^{\dot{\delta}} \mathcal{D}_{\alpha} G_{\beta \dot{\delta}} . \quad (4.18)$$

This leads to

$$\epsilon^{\delta} \mathcal{D}_{\delta} \mathcal{D}_{\alpha} G_{\beta \dot{\beta}}| - \bar{\epsilon}^{\dot{\delta}} \bar{\mathcal{D}}_{\dot{\delta}} \mathcal{D}_{\alpha} G_{\beta \dot{\beta}}| = 0 , \quad (4.19)$$

and hence

$$\mathcal{D}_{\delta} \mathcal{D}_{\alpha} G_{\beta \dot{\beta}}| = \bar{\mathcal{D}}_{\dot{\delta}} \bar{\mathcal{D}}_{\dot{\alpha}} G_{\beta \dot{\beta}}| = 0 , \quad (4.20a)$$

$$\bar{\mathcal{D}}_{\dot{\delta}} \mathcal{D}_{\alpha} G_{\beta \dot{\beta}}| = \mathcal{D}_{\delta} \bar{\mathcal{D}}_{\dot{\alpha}} G_{\beta \dot{\beta}}| = 0 . \quad (4.20b)$$

These conditions⁹ imply, in particular, that $R|G_{\beta \dot{\beta}}| = 0$ and $\mathcal{D}_{\alpha \dot{\gamma}} G_{\beta \dot{\beta}}| = 0$, or equivalently

$$M b_c = 0 , \quad (4.21a)$$

$$\nabla_a b_c = 0 . \quad (4.21b)$$

We conclude that the vector field b_a is covariantly constant. Eq. (4.21a) holds if $M = 0$ or $b_a = 0$.

⁹The complete expression for $\bar{\mathcal{D}}_{(\dot{\alpha}} \mathcal{D}^{(\gamma} G^{\delta)}_{\dot{\beta})}|$ is given in [1]. We do not need it for our analysis.

The last condition to analyze is

$$0 = \delta_K W_{\alpha\beta\gamma} = \xi^D \mathcal{D}_D W_{\alpha\beta\gamma} + 3K^\delta{}_{(\alpha} W_{\beta\gamma)\delta} . \quad (4.22)$$

It leads to $\epsilon^\delta \mathcal{D}_\delta W_{\alpha\beta\gamma}| = 0$, and hence

$$\mathcal{D}_\delta W_{\alpha\beta\gamma}| = 0 . \quad (4.23)$$

In virtue of the Bianchi identity (2.4c), the condition $\mathcal{D}^\gamma W_{\alpha\beta\gamma}| = 0$ automatically holds if (4.21b) is satisfied. The nontrivial part of (4.23) is

$$\mathcal{D}_{(\delta} W_{\alpha\beta\gamma)}| = 0 . \quad (4.24)$$

The complete expression for $\mathcal{D}_{(\delta} W_{\alpha\beta\gamma)}|$ is given in [1]. Since the gravitino is absent, eq. (4.24) tells us that the Weyl tensor is equal to zero,

$$C_{\alpha\beta\gamma\delta} = 0 . \quad (4.25)$$

As a result, the space-time is conformally flat.

The above results can be used to read off the Riemann tensor. For this we compute the bar-projection $[\mathcal{D}_a, \mathcal{D}_b]|$ in two different ways. First of all, we can make use of (4.4) to obtain

$$[\mathcal{D}_a, \mathcal{D}_b]| = [\tilde{\nabla}_a, \tilde{\nabla}_b] . \quad (4.26)$$

The right-hand side has to be expressed in terms of the torsion-free covariant derivatives ∇_a . Direct calculations give

$$\begin{aligned} [\tilde{\nabla}_a, \tilde{\nabla}_b]V_c &= [\nabla_a, \nabla_b]V_c - \frac{2}{3}\varepsilon_{abde}b^d\nabla^e V_c \\ &+ \frac{1}{9}\left\{b_c(b_a\eta_{bd} - b_b\eta_{ad}) - b_d(b_a\eta_{bc} - b_b\eta_{ac}) - b^2(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc})\right\}V^d . \end{aligned} \quad (4.27)$$

On the other hand, we can evaluate $[\mathcal{D}_a, \mathcal{D}_b]|$ by making use of eqs. (2.3d) – (2.3f). This gives

$$\begin{aligned} [\mathcal{D}_a, \mathcal{D}_b]|V_c &= -\frac{2}{3}\varepsilon_{abde}b^d\tilde{\nabla}^e V_c - \frac{1}{9}M\bar{M}(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc})V^d \\ &= -\frac{2}{3}\varepsilon_{abde}b^d\nabla^e V_c - \frac{1}{9}M\bar{M}(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc})V^d \\ &+ \frac{2}{9}\left\{b_c(b_a\eta_{bd} - b_b\eta_{ad}) - b_d(b_a\eta_{bc} - b_b\eta_{ac}) - b^2(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc})\right\}V^d . \end{aligned} \quad (4.28)$$

We end up with the Riemann curvature

$$R_{abcd} = -\frac{1}{9}\left\{b_c(b_a\eta_{bd} - b_b\eta_{ad}) - b_d(b_a\eta_{bc} - b_b\eta_{ac}) - b^2(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc})\right\} - \frac{1}{9}M\bar{M}(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}) . \quad (4.29)$$

The Ricci tensor is

$$R_{ab} = \frac{2}{9}(b_ab_b - b^2\eta_{ab}) - \frac{1}{3}M\bar{M}\eta_{ab} . \quad (4.30)$$

The conditions (4.17b), (4.21a), (4.21b) and (4.25) coincide with those given in [15] without derivation. Our expression for the Ricci tensor (4.30) differs from that given in [15] by an overall sign. This difference is due to the different definitions of the Riemann tensor used in [15] and in the present paper. The superspace techniques make the derivation of the conditions (4.17b), (4.21a), (4.21b), (4.25) and (4.30) almost trivial.

Since \mathcal{M} is conformally flat, the corresponding algebra of conformal Killing supervector fields coincides with that of Minkowski superspace, $\mathfrak{su}(2, 2|1)$.

5 Variant off-shell formulations for supergravity

As is well known, there exist three off-shell formulations for $\mathcal{N} = 1$ supergravity in four dimensions: (i) the old minimal formulation ($n = -1/3$) developed first by Wess and Zumino using superspace techniques [2] and soon after in the component field approach [4];¹⁰ (ii) the new minimal formulation ($n = 0$) developed by Sohnius and West [27];¹¹ (iii) the non-minimal formulation ($n \neq -1/3, 0$) pioneered by Breitenlohner [29], who used superspace techniques, and further developed to its modern form by Siegel and Gates [30, 31]. Breitenlohner's formulation [29] is the oldest off-shell supergravity theory in four dimensions.

Each off-shell formulation for $\mathcal{N} = 1$ supergravity can be realized as a super-Weyl invariant coupling of the old minimal supergravity ($n = -1/3$) to a *scalar* compensator Ψ and its conjugate $\bar{\Psi}$ (if the compensator is complex) [1, 32, 33, 34] with a super-Weyl transformation of the form

$$\delta_\sigma \Psi = -(p\sigma + q\bar{\sigma})\Psi , \quad (5.1)$$

¹⁰The linearized version of old minimal supergravity was constructed in [25, 26].

¹¹The linearized version of new minimal [28] supergravity appeared much earlier than [27].

where p and q are fixed parameters which are determined by the off-shell structure of Ψ . The compensator is assumed to be nowhere vanishing.

In this super-Weyl invariant setting, the supergravity prepotentials include the compensator Ψ and its conjugate. If we are interested in a fixed curved background, the supergravity gauge freedom and the super-Weyl invariance should be fixed in a convenient way to eliminate superfluous degrees of freedom. In particular, the super-Weyl invariance can be used to eliminate some of the component fields of Ψ and its conjugate. The isometries of the resulting curved superspace are generated by those conformal Killing supervector fields $\xi = \xi^B E_B$, eq. (3.1), which leave the compensator invariant, that is

$$\xi^B \mathcal{D}_B \Psi - (p\sigma + q\bar{\sigma})\Psi = 0 \quad \Longleftrightarrow \quad (p\sigma + q\bar{\sigma}) = \xi^B \mathcal{D}_B \ln \Psi . \quad (5.2)$$

5.1 Old minimal supergravity

The compensators in old minimal supergravity are a covariantly chiral scalar Φ , $\bar{\mathcal{D}}_{\dot{\alpha}} \Phi = 0$, and its conjugate $\bar{\Phi}$. The super-Weyl transformation of Φ can conveniently be chosen to be

$$\delta_{\sigma} \Phi = -\sigma \Phi , \quad (5.3)$$

and thus $p = 1$ and $q = 0$.

The super-Weyl gauge freedom can be used to impose the gauge condition

$$\Phi = 1 . \quad (5.4)$$

In this gauge the equation (5.2) becomes

$$\sigma = 0 , \quad (5.5)$$

and therefore the isometries are described by the Killing spinor equation (3.11).

5.2 New minimal supergravity

In new minimal supergravity, the compensator \mathfrak{L} is a real covariantly linear scalar,

$$(\bar{\mathcal{D}}^2 - 4R)\mathfrak{L} = 0 , \quad \bar{\mathfrak{L}} = \mathfrak{L} . \quad (5.6)$$

Such a superfield describes the $\mathcal{N} = 1$ tensor multiplet [35].¹² Its super-Weyl transformation is uniquely determined (see [1] for more details)

$$\delta_\sigma \mathfrak{L} = -(\sigma + \bar{\sigma}) \mathfrak{L} , \quad (5.7)$$

and thus $p = q = 1$.

In new minimal supergravity, the super-Weyl gauge freedom may conveniently be fixed by imposing the conditions

$$R| = 0 , \quad (5.8a)$$

$$\mathfrak{L}| = 1 , \quad (5.8b)$$

$$\mathcal{D}_\alpha \mathfrak{L}| = 0 . \quad (5.8c)$$

This leaves unbroken a $U(1)_R$ gauge symmetry generated by $i(\bar{\sigma} - \sigma)|$. The gauge field for this local symmetry is the auxiliary field b_a , in accordance with eq. (2.8b).

In the gauge (5.8), there still remains a single component field contained in \mathfrak{L} that can be defined as (see, e.g., [37]):

$$-2H_{\alpha\dot{\alpha}} := ([\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}] - 2G_{\alpha\dot{\alpha}})\mathfrak{L}| = [\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}]\mathfrak{L}| + \frac{4}{3}b_{\alpha\dot{\alpha}} . \quad (5.9)$$

Here $H^a = \frac{1}{3!}\varepsilon^{abcd}H_{bcd}$ is the Hodge-dual of the field strength of a gauge two-form. Eq. (5.8a) means that $M = 0$. This auxiliary field is not present in new minimal supergravity.

The Killing equation (5.2) corresponding to new minimal supergravity has the form

$$(\sigma + \bar{\sigma}) = \xi^B \mathcal{D}_B \ln \mathfrak{L} . \quad (5.10)$$

We are in a position to derive an equation for unbroken rigid supersymmetries. All definitions and conditions given at the beginning of section 4 remain intact modulo the fact that $M = 0$ in the case under consideration. From the Killing equation (5.10) we deduce that

$$\zeta_\alpha = -\frac{3}{2}\mathcal{D}_\alpha \sigma| = -\frac{3}{4}\bar{\epsilon}^{\dot{\beta}}[\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\beta}}]\mathfrak{L}| = (b_{\alpha\dot{\beta}} + \frac{3}{2}H_{\alpha\dot{\beta}})\bar{\epsilon}^{\dot{\beta}} , \quad (5.11a)$$

and hence

$$\bar{\zeta}^{\dot{\alpha}} = -(b^{\dot{\alpha}\beta} + \frac{3}{2}H^{\dot{\alpha}\beta})\epsilon_\beta . \quad (5.11b)$$

¹²The compensator for new minimal supergravity is often called the improved tensor multiplet [36], because the corresponding action must be super-Weyl invariant, which corresponds to a uniquely determined self-coupling for \mathfrak{L} , with the superfield Lagrangian being proportional to $\mathfrak{L} \ln \mathfrak{L}$.

Plugging this into (4.10) and setting $M = 0$ gives

$$\mathfrak{D}_a \epsilon_\beta + \frac{i}{2} (\sigma_a \tilde{\sigma}_c \epsilon)_\beta H^c = 0 . \quad (5.12)$$

This equation is invariant under the unbroken $U(1)_R$ gauge group for which b_a is the gauge field and \mathfrak{D}_a the gauge covariant derivative.

Eq. (5.12) coincides with the Killing spinor equation in new minimal supergravity [15, 19, 20]. All results of these papers, which concern supersymmetric backgrounds in new minimal supergravity, follow from this equation. The conditions on the background fields implied by (5.12) may be uncovered by studying the corresponding integrability conditions, see e.g. [21]. Alternatively, one may use the superspace formalism and analyze implications of the relations

$$0 = \xi^D \mathcal{D}_D R - 2\sigma R - \frac{1}{4} (\bar{\mathcal{D}}^2 - 4R) \bar{\sigma} , \quad (5.13a)$$

$$0 = \xi^D \mathcal{D}_D G_{\alpha\dot{\alpha}} + K_\alpha{}^\delta G_{\delta\dot{\alpha}} + \bar{K}_{\dot{\alpha}}{}^\delta G_{\alpha\delta} - \frac{1}{2} (\sigma + \bar{\sigma}) G_{\alpha\dot{\alpha}} + i \mathcal{D}_{\alpha\dot{\alpha}} (\bar{\sigma} - \sigma) , \quad (5.13b)$$

$$0 = \xi^D \mathcal{D}_D W_{\alpha\beta\gamma} + 3K^\delta{}_{(\alpha} W_{\beta\gamma)\delta} - \frac{3}{2} \sigma W_{\alpha\beta\gamma} , \quad (5.13c)$$

in conjunction with the following corollary of (5.10):

$$\begin{aligned} 0 = & \xi^D \mathcal{D}_D \mathfrak{H}_{\alpha\dot{\alpha}} + K_\alpha{}^\delta \mathfrak{H}_{\delta\dot{\alpha}} + \bar{K}_{\dot{\alpha}}{}^\delta \mathfrak{H}_{\alpha\delta} \\ & - \frac{3}{2} (\sigma + \bar{\sigma}) \mathfrak{H}_{\alpha\dot{\alpha}} + \frac{3}{2} (\mathcal{D}_\alpha \sigma) \bar{\mathcal{D}}_{\dot{\alpha}} \mathfrak{L} - \frac{3}{2} (\bar{\mathcal{D}}_{\dot{\alpha}} \bar{\sigma}) \mathcal{D}_\alpha \mathfrak{L} , \end{aligned} \quad (5.14)$$

where we have denoted

$$\mathfrak{H}_{\alpha\dot{\alpha}} := -\frac{1}{2} ([\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}] - 2G_{\alpha\dot{\alpha}}) \mathfrak{L} . \quad (5.15)$$

We will not pursue such an analysis here.

5.3 Non-minimal supergravity

The compensators in non-minimal supergravity are a complex covariantly linear scalar Σ constrained by

$$(\bar{\mathcal{D}}^2 - 4R) \Sigma = 0 , \quad (5.16)$$

and its complex conjugate. The super-Weyl transformation of Σ is not determined uniquely by the constraint,

$$\delta_\sigma \Sigma = \left(\frac{3n-1}{3n+1} \sigma - \bar{\sigma} \right) \Sigma , \quad (5.17)$$

where n is a real parameter such that $n \neq -1/3, 0$. Thus the Killing equation corresponding to non-minimal supergravity is

$$\bar{\sigma} - \frac{3n-1}{3n+1}\sigma = \xi^B \mathcal{D}_B \ln \Sigma . \quad (5.18)$$

As has recently been shown [38], the only way to construct non-minimal supergravity with a cosmological term¹³ is to fix $n = -1$ and consider a compensator Γ obeying the deformed linearity constraint

$$-\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\Gamma = \mu = \text{const} . \quad (5.19)$$

The super-Weyl transformation of Γ is

$$\delta_\sigma \Gamma = (2\sigma - \bar{\sigma})\Gamma . \quad (5.20)$$

It is obtained from (5.17) by replacing $\Sigma \rightarrow \Gamma$ and setting $n = -1$. One may check that the left-hand side of (5.19) is super-Weyl invariant. The Killing equation for this supergravity formulation is obtained from (5.18) by choosing $n = -1$ and replacing $\Sigma \rightarrow \Gamma$. Anti-de Sitter superspace is a maximally symmetric solution of this theory [38].

6 Conclusion and outlook

In this paper we have re-derived some of the key results of [15, 19, 20] from the more general superspace setting developed in [1]. The superspace approach of [1] is more general simply because it takes care of all the isometries of a given curved background, and not just the rigid supersymmetry transformations as in [15, 19, 20]. If one is interested in generating all possible supersymmetric backgrounds in 4D $\mathcal{N} = 1$ off-shell supergravity, the results of [15, 16, 17, 18, 19, 20, 21, 22, 23] appear to be exhaustive. However, if the goal is to engineer off-shell rigid supersymmetric theories on a given curved spacetime, or to carry out supergraph calculations in such theories, the superspace symmetry formalism of [1] (and its extensions) is most powerful. In this respect, it is worth mentioning the explicit construction of the most general 4D $\mathcal{N} = 2$ supersymmetric sigma models in anti-de Sitter space [11].

Old minimal supergravity with the super-Weyl invariance can be thought of as $\mathcal{N} = 1$ conformal supergravity. From this point of view, the super-Weyl invariant approach to all

¹³Non-minimal anti-de Sitter supergravity was argued in [40] not to exist. This is indeed so if one deals with the standard constraint (5.16).

known off-shell $\mathcal{N} = 1$ supergravity theories, which we sketched in section 5, is simply a version of the general principle that Poincaré (super)gravity can be realized as conformal (super)gravity coupled to certain compensators [39] (see also [36]). This principle is universal, for it also applies to extended supergravities in four dimensions and, more generally, to supergravity theories in diverse dimensions. Therefore, our approach to the symmetries of curved 4D $\mathcal{N} = 1$ superspace backgrounds can readily be extended to supergravity theories in diverse dimensions. Suitable superspace formulations were constructed in [41] for 4D $\mathcal{N} = 2$ supergravity, [42] for 5D $\mathcal{N} = 1$ supergravity, [43] for 3D \mathcal{N} -extended supergravity, [44] for 6D $\mathcal{N} = (1, 0)$ supergravity. It is an interesting open problem to study supersymmetric backgrounds supported by these supergravity theories using the superspace techniques.

Regarding the $\mathcal{N} = 1$ case in four dimensions studied in the present paper, there exist alternative superspace formulations for conformal supergravity developed by Howe [45] (see [40] for a review) and Butter [46], which are characterized by larger structure groups than the Lorentz group $\text{SL}(2, \mathbb{C})$ characteristic of the Wes–Zumino superspace geometry.¹⁴ It would be interesting to make use of these formulations to study supersymmetric backgrounds. In particular, since the structure group in Howe’s formulation is $\text{SL}(2, \mathbb{C}) \times \text{U}(1)_R$, this approach is most suitable to describe new minimal supergravity.

While this paper was in the process of writing up, there appeared a preprint [47] devoted to the construction of supersymmetric backgrounds associated with one of the off-shell formulations for 3D $\mathcal{N} = 2$ supergravity developed in [12, 43]. The analysis in [47] is purely component. A superspace construction may be developed along the lines described in the present paper. In fact, the supersymmetric backgrounds allowing four supercharges were constructed in [12] much earlier than [47] using purely superspace tools.

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¹⁴The formulation for $\mathcal{N} = 1$ conformal supergravity given in [46] may be viewed as a master one in the sense that all other formulations can be obtained from it by partial gauge fixings, see [38] for a review.

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